

RATIONAL COMMENSURABILITY & WHY DIVISION BY ZERO IS NOT POSSIBLE

PREFACE

To the diligent reader

A flood of notes on false theorems is tautology, and is painstakingly hard to prove the intuition behind those rules. In this case, its rather obnoxious to memorize theorems that are barely clear if you read too much into them, or in their rules for that matter. Nonetheless, the principles when clearly understood, demonstrate fully, the properties of rigour in solving functions.

Mathematicians, if the title is deserved, should not be lazy. Every concept must be accessible to all readers. This implies, explanation of procedures, are very well structured, and with clarity. If the point is not in sight, then its highly probable the material being addressed lacks originality, or even a mathematical mind. At this point, you can close the book, and look for an alternative text entirely, or otherwise bare the exhaustion from inadequate and confusing space of words and equations.

In this text, we'll interpret the Algebra of Time; from perception, to perspective, and work out solutions to clarify the problems. If the concept is then understood, it will be easier to obtain, or write down the entire structure of a DEFINITION. Therefore, to understand most PROOFS, we'll make comparisons of properties of functions, which are DEFINITELY NUMERICAL. We'll then analyze the ALGEBRAIC composition that lead to the results of a well structured idea of RATIONAL COMMENSURABILITY. The critical basis of the form of this text, is consistency, and traced down to first principles, so that the logical nature in the subject of Mathematics is complete, self-contained, and a logical presentation that is well established throughout.

SECTION I: MATHEMATICAL MODELS

1.The Concept of Calculus

From the Pythagorean Theorem, the Quadratic Formula, and the formulae for calculating Surface Areas, Volumes of Cones and Spheres, we can define DIFFERENTIATION, as the PROCESS of finding DERIVATIVES and ANTIDERIVATIVES. For instance, the GRAVITY FUNCTION, is understood in the form of:

$$a[t] = \partial_{[t]} \left\{ \frac{v}{t} \right\} = \pm 9.8 \, m/s^2 \text{ [ACCELERATION]}$$

$$v(t) = \partial_{[t]} \left\{ \frac{s}{t} \right\} = 9.8 \, t + C \text{ [VELOCITY/SPEED] and}$$

$$s(t) = 4.9 \, t^2 + Ct + D \text{ [POSITION/DISTANCE]}$$

The PRINCIPLE OF DIFFERENTIATION, here, is to derive the value of the CONSTANT in a given DERIVATIVE to obtain its FUNCTION. This can be rewritten in the form where:

$$f''(x) = 12(x^2) \quad \text{[2nd Derivative]}$$

(4)

$$f'(x) = 4(x^3) + C \quad \text{[1st Derivative]}$$

(5)

$$f(x) = x^4 + C(x) + D$$

Thus, the quadratic formula, is solely used to solve the equations of DEGREE 2, written as :

$$s[t] = 4.9 \, t^2 + Ct + D$$

(6)

which is the POSITION FUNCTION, of MOTION.

$$\text{If } f' = x^3, \text{ then } f = 3[x^2]; \text{ and if } f = -4x, \text{ then } f = -2[x^2]$$

Let $f(x) = -\cos(x)$, and $g(x) = -\cos(x) + C$; then $f(x) = \sin(x)$ and $g'(x) = \sin(x)$. So f' and g' are ANTIDERIVATIVES of f .

The QUOTIENT $\frac{a}{b}$ can be interpreted to represent the y-coordinate and the x-coordinate where (a), is the horizontal ASYMPTOTE, and (b) is the vertical asymptote.

We shall demonstrate the use of DIFFERENTIALS, to estimate ERRORS, that occur in APPROXIMATE MEASUREMENTS; that is, RELATIVE or PERCENT-AGE ERROR.

We know that (e), is the base of the NATURAL LOGARITHM. So the $f \ln(x) = f[\ln(x)]$ when the base $[a]=[e]$. This function should imply the concept of an ABSOLUTE VALUE.

When solving COMPOSITE FUNCTIONS, it is understood that $\frac{\partial(y)}{\partial(x)} = \frac{1}{u} \frac{\partial(u)}{\partial(x)}$

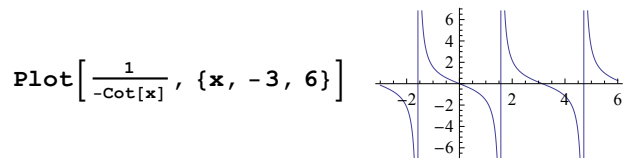
and with LOGARITHMIC LAWS, we obtain the derivatives of TRIGONOMETRIC FUNCTION, which are PATTERNS, and then, we deduce that the SIGNS go with the COFUNCTIONS.

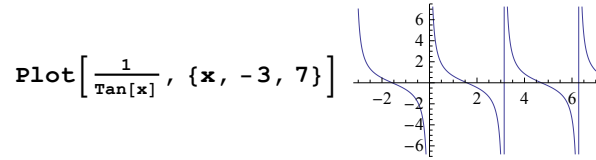
Let us ILLUSTRATE a METHOD in the ANALYSIS of PATTERNS of TRIGONOMETRIC DERIVATIVES, by showing a PROOF, that the function $f \{ \cot(x) \} = \tan(x)$.

First, let's analyze some deductive algebraic operation: The $\sqrt{4}$ can be rewritten as $\frac{1}{2}[4]^{-\frac{1}{2}}$; $2^2 = 4$; $\log_{10}(4) = 2$. We then use this notion, to construct our proof. In the following table, we look for a pattern.

TRIGONOMETRIC	DERIVATIVES	TRIGONOMETRIC TABLE	PROOF
$f' \sin(x) = \cos(x)$	$f' \csc[x] = -\csc[x] \cot[x]$	$\frac{\sin[x]}{\cos[x]} = \tan[x]$	$\tan[x] = \sec^2[x]$
$f' \cos(x) = -\sin(x)$	$f' \sec[x] = \sec[x] \tan[x]$	$\frac{\cos[x]}{-\sin[x]} = \sec^2[x]$	$\cot[x] = -\csc^2[x]$
$f' \tan(x) = \sec^2(x)$	$f' \cot[x] = -\csc^2[x]$	$\frac{\csc[x]}{\sec[x]} = \cot[x]$	
		$\frac{-\csc[x] \cot[x]}{\sec[x] \tan[x]} = -\csc^2[x]$	

From the table, we MUST see that $\csc[x] = \sec[x]$ and $-\csc[x] \cot[x] = \sec[x] \tan[x]$. It follows that $\frac{\csc[x]}{-\csc[x] \cot[x]}$ can be rewritten as $\frac{1}{-\cot[x]}$. Now, $\sqrt{x}$ can be expressed as $\frac{1}{2}[x]^{-\frac{1}{2}}$ or $\frac{\log[x]}{\log[2]}$. Therefore, the INVERSE of $\frac{1}{-\cot(x)} = \tan(x)$; $\frac{\sec[x]}{\sec[x] \tan[x]}$ can be rewritten as $\frac{1}{\tan[x]}$. Therefore, the INVERSE of $\frac{1}{\tan[x]} = \cot(x)$. Thus, solving the QUOTIENT on the left-hand side, we get $\tan[x] = \cot[x]$. Have a look at the CURVES, in the respective GRAPHS.





In summary, at the very core of Mathematics, is its highly developed SYMBOLIC LANGUAGE, that unifies a broad range of the PROGRAMMING concept. We'll adhere to this paradigm, throughout the text.

2. Linear Approximation[The Derivative]

Consider the function $L(x) = f(a) + f'(a)[x - a]$. Let us then, APPROXIMATE a TANGENT LINE, for this function satisfying the following conditions:

I: $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty) \rightarrow$ 1st Derivative

II: $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2) \rightarrow$ 2nd Derivative

III: $\lim_{x \rightarrow \infty} f(x) = -2$, and $\lim_{x \rightarrow \infty} f(x) = 0$ to represent two horizontal ASYMPTOTES at $(y) = -2$ and where $(y) = 0$ is the x - axis.

While *sketching the curve*, we note that: at the points $(x) < -2$ and $(x) > 2$, the *curve bends(concave) upwards*; for $-2 \leq [x] \leq 2$ the *curve bends(concave) downwards*. The analysis enables us to determine the INFLECTION POINTS, of which, are the MAXIMUM VALUES of the curve.

Its useful to keep in mind that the derivative of the ACCELERATION FUNCTION(1.1), written as $\frac{\partial a}{\partial t} = \frac{\partial^3 s}{\partial t^3}$ is defined to be the RATE OF CHANGE of $[a]$.

This called JERK.

Thus, the DIFFERENTIATION OPERATION where f is the VELOCITY function($v[t]$), then f' is the ACCELERATION function($a[t]$), and is said to be the 2nd derivative of POSITION function($s[t]$), written as $\frac{\partial^2 s}{\partial t^2}$.

Therefore, the DERIVATIVE, in the case of the *MOTION OF A PARTICLE*, is defined to be the RATE OF CHANGE. Thus, the VELOCITY of a particle at $(t) = (a)$, the SPEED is the $|f'(a)|$.

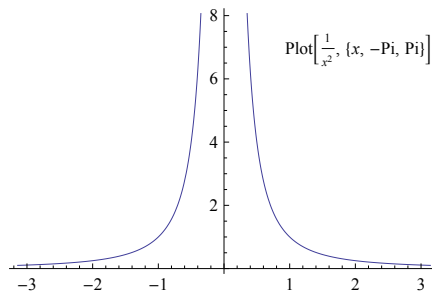
At this point, we can write the GENERAL NOTATION for the derivative as $f'(x) = \frac{\partial[y]}{\partial[x]} = \lim_{h \rightarrow 0} f\left(\frac{(h+x)-f(x)}{h}\right)$. This is the SLOPE, of the TANGENT LINE at the point $P[(x_1), f(x_1)]$.

So, if the function $y = f(x)$, then the derivative $y' = \frac{\partial[y]}{\partial[x]}$ is interpreted as the rate of change of (y) with respect to (x).

Here are the common alternative notations for the derivative[also called OPERATORS]: y' , $\frac{\partial}{\partial[x]}$, $\frac{\partial[y]}{\partial[x]}$, $\frac{\partial}{\partial[x]} f(x)$, $D f(x)$ and $D f(x)$.

★ The power of Mathematics, lies in its ABSTRACTNESS. Thus, the derivative concept, can have different interpretations/analogies.

3. Full Proof Methods For Calculating Limits



The $\lim_{x \rightarrow \frac{\pi}{2}} \tan(x) = \infty$ has the vertical asymptote at $[x] = 0$. From the graph, the VALUES of the curve do not APPROACH[$\rightarrow$] a FIXED NUMBER. This means, as the values of $[x] \rightarrow 0$, the values of $[y] \rightarrow \infty$. Thus, we say that the $\lim_{x \rightarrow 0} \left(\frac{1}{x^2}\right)$ DOES NOT EXIT. We use this concept of the LIMIT to define the

PROPERTIES OF A FUNCTION.

We can also apply the concept of the ABSOLUTE VALUE, to describe the LIMIT. For instance,

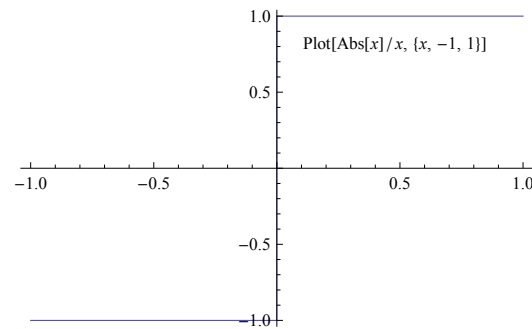
the $|[x]| = [x]$ if $[x] \geq 0$

$= -[x]$ if $[x] < 0$. Thus, the HEAVISIDE FUNCTION, $H[t] = 0$ if $[t] < 0$

$= 1$ if $[t] > 0$ can be used to describe CURRENT that is switched on at time $[t]$. So, there is

no single number that $H[t]$ approaches as $[t] \rightarrow 0$. Thus, the Limit DOES NOT EXIST. To clarify what this means, observe the GRAPH of the function,

$$f[x] = \frac{|x|}{x}$$



As $[x] \rightarrow 0$, there is an OSCILLATION between 1 and -1.

Therefore, for a function $f[x] = 0$, we can obtain the ROOTS, which is the VALUE of the LIMIT. In other words, there exists a NUMBER $[x]$ in the INTERVAL $[a, b]$, such that $f[x] = 0$

4. Limit of A Function

If the $\lim_{x \rightarrow a} (g[x]) = [b]$, then $f(g[x]) = [b]$. This means, if $[x]$ is close to $[a]$, then $g[x]$ is close to $[b]$; and $f(g[x])$ is close to $f[b]$. Thus, $f[g]$ is called a COMPOSITE FUNCTION.

We use a Computer Algebra System, or a Graphing Calculator to construct our functions. For instance, The $\lim_{t \rightarrow 0} \frac{\sqrt{t^2+9}-3}{t^2} = \frac{1}{6}$. The function is the ARGUMENT, while $\frac{1}{6}$ is the VALUE.

To obtain the 1st derivative of a function f' , we use the formula $f[x^n] = n[x^{n-1}]$ and its called the POWER RULE.. It follows that we can find POINTS ON A CURVE given the TANGENT LINE. and so we DIFFERENTIATE to obtain the ROOTS.

Let $S = 2t^3 - 5t^2 - 3t + 4$. We can interpret this function as the Motion of a Particle where S is in centimeters [cm] and the unit of $[t]$ is in seconds (s). Therefore, the 1st devirative of the function using the POWER RULE is: $f' = 6t^2 - 10t - 3$ (This is the Instantaneous Rate of Change/Displacement/Velocity of the particle); and the 2nd derivative is;

$f'' = 12t - 10$ (which is the Acceleration) We can thus evaluate when $[t] = 2$, $[a] = 14 \text{ cm/s}^2$

Let us solve the INEQUALITY when the $|x - 3| + |x + 2| < 11$. First, note that;

$|x| = [x]$ if $[x] \geq 0$

$= -[x]$ if $[x] < 0$. It follows that, $|x - 3| = [x - 3]$ if $[x - 3] \geq 0$, $= [x - 3]$ if $[x] \geq 3$

$= -[x - 3]$ if $[x - 3] < 0$, $= -[x + 3]$ if $[x] < 3$

$|x + 2| = [x + 2]$ if $[x + 2] \geq 0$, $= [x + 2]$ if $[x] \geq -2$

$= -[x + 2]$ if $[x + 2] < 0$, $= -[x - 2]$ if $[x] < -2$

These expressions show that we MUST consider three cases: when $[x] < -2$, $-2 \leq [x] < 3$, and $[x] \geq 3$.

Case I. If $[x] < -2$, we have $-x - 3 - x - 2 < 11$. So $-2x - 5 < 11$. Solving for x , we obtain $[x] < -5$.

Case II. If $-2 \leq [x] < 3$, we have $-x + 3 + x + 2 < 11$. Solving the inequality, we obtain $5 < 11$.

Case III. If $[x] \geq 3$, we have $x - 3 - x + 2 < 11$. So $2x < 12$. Solving for x , we obtain $[x] < 6$

We then combine the three cases and write that the inequality is satisfied when $-5 < [x] < 6$. Therefore, the SOLUTION is the ROOTS/DOMAIN/INTERVAL $[-5, 6]$ on the x - axis. We can then interpret $y = f[x]$ as a FUNCTION, and $f'[x]$ its DERIVATIVE, which is equivalent to $\frac{\partial[y]}{\partial[x]} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \partial[y]}{\partial[x]}$, which is the

RATE OF CHANGE of $[y]$ with respect to $[x]$, and is denoted by the symbol (Δ) .

In **DIFFERENTIAL CALCULUS**, we use the TANGENT LINE $[m] = \lim_{Q \rightarrow P} [PQ] = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ to approximate RATES OF CHANGE.

In **INTEGRAL CALCULUS**, we 'll use the $\lim_{n \rightarrow \infty} [A_n] = A$ to approximate AREAS.

★ Keep in mind the notion that Differential and Integral Calculus are INVERSE OPERATIONS.

5. Limit Of A Sequence

If $a_1 = (3.1 \dots \dots \dots \dots \dots \dots \dots \dots \dots a_7 = 3.1415926$, then the $\lim_{n \rightarrow \infty} [a_n] = \pi$. This means the RATIONAL APPROXIMATION to π . We can then say that the Limit of a Sequence is 0 implies $\lim_{n \rightarrow \infty} \left[\frac{1}{n} \right] = 0$. Thus, we can define the GENERAL NOTATION for the limit of a sequence as $\lim_{n \rightarrow \infty} [a_n] = L$.

We can also say that the $\lim_{n \rightarrow \infty} [a_n] = P = \lim_{n \rightarrow \infty} [t_n]$ where $a_n < t_n$ for all $[n]$, is a SEQUENCE WITH THE SAME LIMIT.

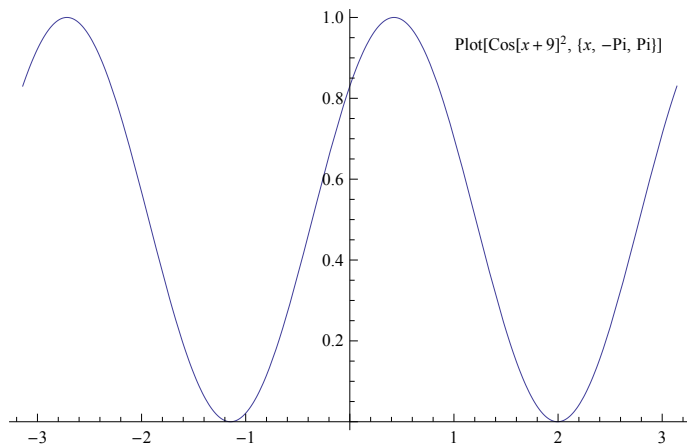
So, the $\lim_{n \rightarrow \infty} [S_n] = 1$, can be interpreted as the SUM OF A SERIES.

Consider the RATIONAL FUNCTION $f(x) = x^{-1} = \frac{1}{x}$. We can thus define the PROPERTY OF AN INCREASING/DECREASING FUNCTION as a RECIPROCAL FUNCTION. For instance, BOYLE's LAW is the function $V = \frac{c}{P}$ where $\frac{c}{P}$ is a RATIONAL NUMBER.

6. Types Of Functions

For the function $f[x] = \text{Cos}^2[x + 9]$ is interpreted as; add 9, take the Cosine of the result, then square.

If $f = f[g(h)]$, then $h[x] = [x + 9]$, $g[x] = \text{Cos}[x]$, and $f[x] = [x^2]$. Thus, $[f[g[h][x] = f[g(x + 9)] = f[\text{Cos}(x + 9)] = [\text{Cos}(x + 9)]^2 = f(x)$ is the DECOMPOSITION of the TRIGONOMETRIC function. The order of operation is illustrated from the graph below:



If $f[x] = [y]$, then $f^{-1}[y] = [x]$. This is an example of an INVERSE FUNCTION. It implies a REFLECTION about $[y]=[x]$.

For instance, $\log_{10}(0.001) = -3$; So $10^{-3} = 0.001$. Thus, the general notation for the LOGARITHMIC FUNCTION is $\log_a[x] = [y]$ implies $[a]^y = [x]$.

We've already seen examples of the POWER(RULE), COMPOSITE, RECIPROCAL/RATIONAL and TRIGONOMETRIC functions. We'll later build on the definitions of TRANSCENDENTAL, EXPONENTIAL and PARAMETRIC CURVES.

The STRUCTURE OF CALCULUS is built largely from the BINOMIAL FORMULA. It is defined as $P[x] = a_n x^n + a_{n-1} x^{n-1} + a_2 x^2 + a_1 x^1 + a_0$ where $[n]$ is the DEGREE of the POLYNOMIAL. This is the core and routine of many COMPUTER ALGEBRA COMPUTATIONS in the form of ALGEBRAIC FUNC-

TIONS where $f[x] = 0$.

Now that we've understood the concept of the DERIVATIVE, in the proceeding chapters, we'll discuss the process of combining DIFFERENTIAL and INTEGRAL CALCULUS.

7. Numerical Integration

Let us demonstrate a simple ALGORITHM for numerical integration. Below is a table of comparison for the $\int_0^1 \frac{1}{x} dx$ for Right-Hand End-Points.

ERROR	F	n ≤ K
E_M	f''	41
E_T	f'	29
E_S	$f^{[4]}$	8

The table is the result on ACCURACY for T_n , M_n , and S_n to within $K = 0.0001$ for the $\int_0^1 \frac{1}{x} dx$.

$$S_n = \int_0^1 \frac{1}{x} dx = \frac{\Delta[x]}{3} [f[x] + 4f[x_1] + 2f[x_2] + \dots + 2f[x_{n-2}] + 4f[x_{n-1}] + f[x_n]] \text{ where } [n] \text{ is EVEN and } \Delta[x] = \frac{[b-a]}{n}.$$

First observe the PATTERNS OF COEFFICIENTS for S_n is 1, 4, 2, 4, 2,, 4, 2, 4, 1. The RULE is a WEIGHTED AVERAGE of T_n and M_n . If $f^{[4]}[x] = \frac{24}{[x^3]}$,

$$\text{then } |f^{[4]}[x]| = \frac{K[b-a]^4}{180[n^4]} = |E_s|. \text{ Thus, } S_{2[n]} = \frac{1}{3} [T_n] + \frac{2}{3} [M_n].$$

The objective is to find $[n]$ for $T_{[n]}$ and $M_{[n]}$ for a given value of K , where $n \leq K$ is the desired result. This is a method for estimating the ACCURACY OF APPROXIMATION using the ERROR BOUND FORMULA for R_n , L_n , M_n , and T_n where $T_n[\text{TRAPEZOIDAL RULE}] = \frac{1}{2} [L_n + R_n]$.

$$T_n = \int_a^b \frac{1}{x} f[x] dx = \frac{\Delta[x]}{2} [f[x_0] + f[x_1] + f[x_1] + f[x_2] + \dots \dots \dots + f[x_{i-n}] + f[x_n]] \text{ (These are the INTERVALS)}$$

$$= \frac{\Delta[x]}{2} [f[x_0] + 2f[x_1] + 2f[x_2] + \dots \dots \dots + 2f[x_{i-n}] + f[x_n]] \text{ where } \Delta[x] = \frac{[b-a]}{n} \text{ and } [x_i] = [a] + [i] \Delta[x]. \text{ Note that these are patterns of coefficients for NUMERICAL ANALYSIS.}$$

So, for $|f'[x]| \leq K \frac{[b-a]}{12[n^2]}$ and $|f''[x]| \leq K \frac{[b-a]}{24[n^2]}$. We then solve for $[n]$, to get an accuracy to within the constant K .

Observe that the ESTIMATE depends on the size of the 2nd DERIVATIVE. That is, if we let $f[x] = \frac{1}{[x]}$, then $f'[x] = \frac{1}{-[x^2]}$, $f''[x] = \frac{2}{[x^3]}$, and $f^{[4]}[x] = \frac{24}{[x^5]}$ where $f[x]$ represents the INTEGRAL $(\int_0^1 \frac{1}{x} dx)$ and $f'[x] = E_T$, $f^{[1]}[x] = E_M$, and $f^{[4]}[x] = E_S$.

The $|E_T|$, $|E_M|$, and $|E_S|$ are the ERROR APPROXIMATIONS using the ERROR BOUND FORMULA. So we let $E_T = f[x] - T_{[n]}$, $E_M = f[x] - M_{[n]}$, and $E_S = f[x] - S_{[n]}$. If we then substitute the *error bound formula* for E_T , E_M , and E_S , we have $|E_T| = K \frac{[b-a]}{12[n^2]}$, $|E_M| = K \frac{[b-a]}{24[n^2]}$ and the $|E_S| = K \frac{[b-a]}{180[n^2]}$ when $f[x] \leq [a]$, for $[a] \leq [x] \leq [b]$.

For $R_n = \sum_{i=1}^n f(x) \Delta(x)$, $L_n = \sum_{i=1}^n f(x_{n-1}) \Delta(x)$, $T_n = \frac{1}{2}[L_n + R_n]$, and $S_{2(n)} = \frac{1}{3} T_n \frac{2}{3} M_n$, the comparisons indicate that $S_{2(n)}$, gives a better APPROXIMATION for $\int_0^1 \frac{1}{x} dx$ with the use of the ERROR BOUND FORMULA when $\int_0^1 \frac{1}{x} dx$ is CONTINUOUS for the RIGHT-HAND END-POINTS.

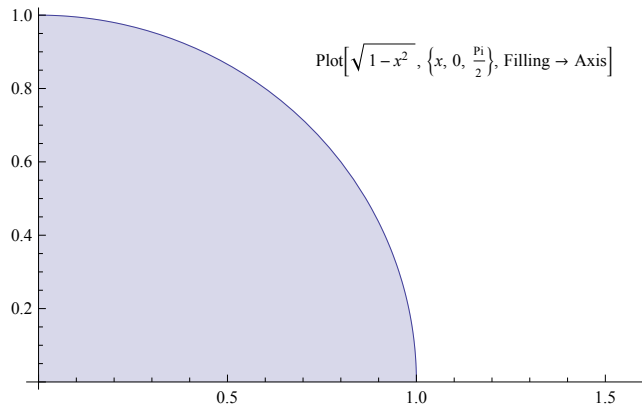
So far, we then define $\int_0^1 e^{[x^2]} dx$ as an ELEMENTARY FUNCTION/INDEFINITE INTEGRAL because it does not have ELEMENTARY ANTIDERIVATIVES.

☞ In the following chapters, we'll construct the the notion of a CONTINUOUS FUNCTION, and the CONCEPT OF AN ANTIDERIVATIVE.

8. Functions With Roots

To eliminate functions with roots, we use powers of TRIGONOMETRIC IDENTITIES and their HALF-ANGLE IDENTITIES. For insatnce $\sqrt{1-x^2}$, we match a considered trigonometric identity $\cos^2[x] = \frac{1}{2} [1 + \cos 2[x]]$ to eliminate the root of the function of the UNIT CIRCLE $[x^2] + [y^2] = 1$.

$[x^2] + [y^2] = 1$ can be rewritten as $[y^2] = (1 - [x^2])$. which is EVEN FUNCTION. IF WE LET $[y] = f[x]$, the $[y] = \sqrt{1-x^2}$. So $f[x] = f[\sqrt{1-x^2}] = f[x] (1 + \cos 2[x])$.



So given the graph above, we can demonstrate the PROOF that $[A]$, of the circle with radius (r) , is equivalent to $\pi(r^2)$. In other words, $[A] = \pi(r^2)$.

First, let $\int_a^b f(x) dx = F = f[x]$, $[A] = \pi(r^2)$ and $[x] = \sin\left[\frac{\pi}{2}\right] = 1$. Then $f[x] = F'[x] = F$

$\frac{1}{4}[A] = r^2 \int_0^{\frac{\pi}{2}} (\cos^2 \theta) d\theta = \frac{1}{2} r^2 \int_0^{\frac{\pi}{2}} (1 + \cos 2[\theta]) d\theta$ where the INTERVAL $[x] = [b, a] = \left[\frac{\pi}{2}, 0\right] = \theta$.

So $\frac{1}{2} r^2 \int_0^{\frac{\pi}{2}} (1 + \cos 2[\theta]) d\theta = \frac{1}{2} r^2 \int_0^{\frac{\pi}{2}} \left(\theta + \frac{1}{2} \sin 2[\theta]\right) = \frac{1}{2} r^2 \left\{2\left(\frac{\pi}{2} + \frac{1}{2} \sin\left[\frac{\pi}{2}\right]\right)\right\} - \{0\}$. We then solve to obtain $\frac{1}{2} r^2 \left\{\left(\frac{\pi}{2}\right)\right\} - \{0\} = \frac{r^2 \pi}{4} = \frac{1}{4}[A]$

Thus, $f[\cos^2 \theta] = F'[1 + \cos 2\theta]$ (the trigonometric identity) $= F\left[\frac{1}{2} \sin 2\theta\right]$ (the half - angle identity) $= f\left(\sqrt{1-x^2}\right) = f[x]$. This is the full proof that $f[x] = F'[x] = F$.

★ If a function $(f[x])$ is equivalent to the Integral $\left(\int_a^b f(x) dx\right)$, then the ANTIDERIVATIVE $(F \text{ or } F')$ is equivalent to the function.

9. The Proof is in The Limit

If we expand $\frac{\partial}{\partial x} f(x) g(x)$, we obtain $f'(x) g(x) + g'(x) f(x)$. If we Integrate;

then $\int f'(x) g(x) + \int g'(x) f(x) = f(x) g(x)$.

We can rearrange the integral to obtain $\int f'(x) g(x) = f(x) g(x) - \int g'(x) f(x)$.

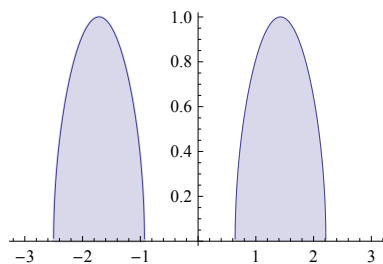
Let $f(x) = u$ and $g(x) = v$ then

$f'(x) = du$ and $g'(x) = dv$. we then substitute the integral we've rearranged to obtain :

$\int (u) dv = (u)(v) - \int (v) du$. The calculation now looks simpler.

On the interval $[-a, a]$, when $f(-x) = f(x)$, then f is EVEN, and when $f(-x) = -f(x)$, f is ODD. This is the definition of the SYMMETRY OF A FUNCTION.

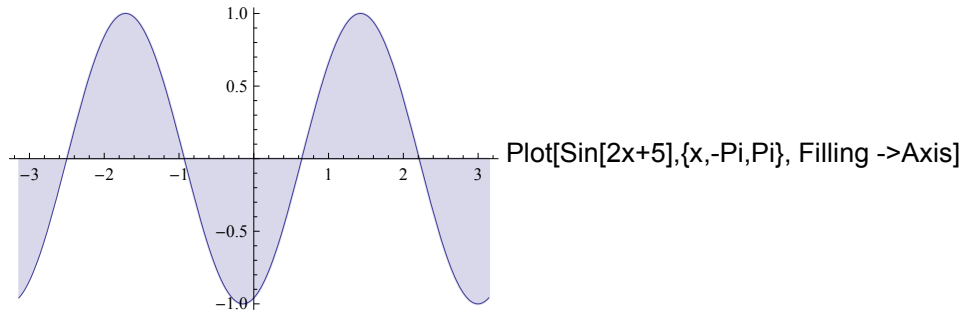
Thus $f(x)$ is EVEN when $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$



`Plot[Sqrt[Sin[2x+5]],{x,-Pi,Pi},Filling->Axis]`

and

$f(x)$ is ODD when $\int_{-a}^a f(x) dx = 0$



★ Observe that the Square Root of any function is EVEN, otherwise its ODD.

Let $u = g(x)$. If $\frac{\partial}{\partial x} = \frac{\partial[y]}{\partial[u]} \frac{\partial[u]}{\partial[x]} = \frac{1}{u} \frac{\partial[y]}{\partial[u]}$, then $\int f[g(x) g'(x)] dx = \int (u) d(u)$ because $g(x) = u$ and $g'(x) = du$

Thus we know that $g'(x) = F'[x] = F$. So we can also say $\int_{g[a]}^{g[b]} f(u) du = F'[u] \int_{g[a]}^{g[b]}$. Therefore, $\partial[x]$ and $\partial[u]$ are DIFFERENTIALS.

If g' is continuous on $[a,b]$, and F is continuous on the RANGE of $(u) = g(x)$, then $\int_{[a]}^{[b]} f[g(x) g'(x)] \partial[x] = \int_{g[a]}^{g[b]} f(u) du = F'[u] \int_{g[a]}^{g[b]} = F[g(b)] - F[g(a)]$.

We then write that $\int_{[a]}^{[b]} f(x) \partial[x] = F[b] - F[a] = F' \int_{[a]}^{[b]} = \lim_{x \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta(x)$ where $F' = F$.

Note that (x_i^*) is chosen to be the MIDPOINT $[\bar{x}_i]$ of the SUBINTERVAL $[x_{i-1}^*, x_i]$. This is the midpoint approximation of $M_{[n]}$ where $[\bar{x}_i] = \frac{1}{2} [x_{i-1} + x_i]$ or the MIDPOINT $[x_{i-1}, x_i] = \bar{x}_i = x_i^*$.

★ It is thus clear from chapter [8] that if $g(x) = \int_a^b f(x) dx$, then $F'[x] = f(x)$. This is the concept of the ANTIDERIVATIVE.

Application Of The Limit

Let us clarify the Limit Concept.

The $\int_{t_1}^{t_2} v(t) dt = S'(t) \int_{t_1}^{t_2} = S(t_2) - S(t_1) = \text{DISPLACEMENT}$ and $|v(t)| = \text{DISTANCE/SPEED}$.

So now, $P(t) = E'(t) \int_{t_0}^{t_{24}} = E(t_{24}) - E(t_0) = \text{ENERGY}$ where P is POWER[The Rate of Change of Energy], E is ENERGY and (t) is TIME in HOURS.. Therefore, the $\int_{t_0}^{t_{24}} P(t) dt = \lim_{n \rightarrow \infty} \sum_{i=1}^n P(t_i^*) \Delta(t)$ where $\Delta(t) \left(\frac{24-0}{12} \right) = 2 \text{ HOURS}$.

Thus, from t_0 to t_{24} (written as $t_0 \rightarrow t_{24}$), $R_n = 2[P(1) + P(3) + P(5) + \dots + P(21) + P(23)]$. The TOTAL RATE OF CHANGE is therefore the POWER used on the INTERVAL $0 \leq P \leq 24$ MEASURED in MW-HRS, since $E'(t) = P(t)$.

★ Power is the MEASURE of Energy in Unit Time. Thus, Power(Time) = MegaWatt(Hours)

10. Case of Logic, Followed by Intuition

So far, we know that $F'[x] = f(x)$. Thus, we can define an ANTIDERIVATIVE in the form:

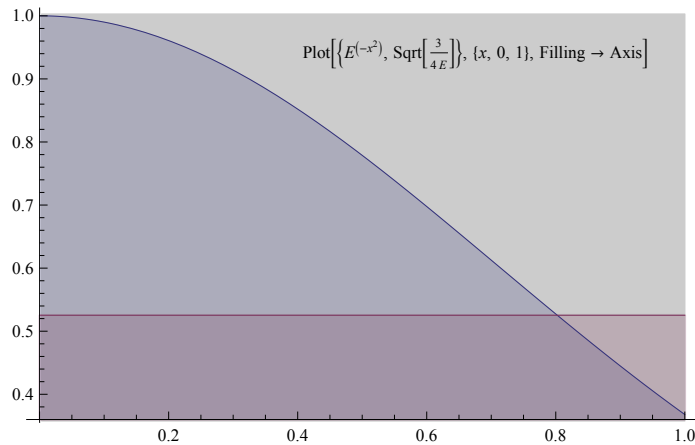
$f(x) = [x^2]$ and $F'[x] = \frac{1}{3}[x^3]$. F' is the antiderivative of $f(x)$, then $\frac{1}{3}[x^3] = [x^2]$.

For instance, the $\int_{[0]}^{[1]} [x^2] dx$, we find that if $f(x) = [x^2]$ then $F'[x] = \frac{1}{3}[x^3]$ where $[x] = [b, a] = [1, 0]$. We then rewrite the integral in the form $\int_{[0]}^{[1]} [x^2] dx =$

$F[b] - F[a]$ where $F = F'$. If we substitute for $[x = 1]$, we obtain $\frac{1}{3}[1^3]1 - \frac{1}{3}[1^3]0 = \frac{1}{3} - 0 = \frac{1}{3}$. Note that the VALUE $\frac{1}{3}$, is the LIMIT OF THE SUM.

Therefore, $F[b] - F[a] = \int_a^b f[x] \partial[x] = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta(x)$. At this point of the construction, we can CALCULATE INTEGRALS by first obtaining the ANTIDERIVATIVE F' . Keep in mind that $f[x] = a_n x^n + a_{n-1} x^{n-1} + a_2 x^2 + a_1 x^1 + a_0 = 0$. To obtain the DERIVATIVE, we use the formula $\left[\frac{\partial}{\partial x}\right] = n[x^{n-1}]$, and for the ANTIDERIVATIVE, we use the formula $F' = \frac{(x^{n+1})}{n+1}$.

Let us now ESTIMATE the $\int_0^1 e^{-[x^2]} \partial[x]$ on the interval $[0,1]$. First, we know that $f(x) = [e^{-[x^2]}]$ since $y = f(x)$. Let $[M] = \text{MAXIMUM VALUE}$ and $[m] = \text{MINIMUM VALUE}$. So we can evaluate the function when $[x] = 0$ by substituting $f(0) = [e^{-[0^2]}] = e = M$, and when $[x] = 1$ as $f(1) = [e^{-[1^2]}] = e^{-1} = \frac{1}{e} = m$. Since $e = 2.7182818$ (called the BASE OF THE NATURAL LOGARITHM), $f(0) = 2.7182818 = e = M$ and $f(1) = [e^{-[1^2]}] = 2.7182818^{-1} = \frac{1}{2.7182818} = m$. We can now graph the result as:



On the graph, $y = e^{-[x^2]}$ is the line of the Curve and is a DECREASING FUNCTION, while $y = \frac{1}{e}$ is the horizontal ASYMPTOTE (Constant)

So, the Integral $\int_0^1 e^{-[x^2]} \partial[x]$ is GREATER than the AREA of the lower RECTANGLE, but LESS than the AREA of the SQUARE.

★ Keep in mind that the DERIVATIVE OF A CONSTANT is 0.

11. Sigma Notation

The FORMULA for SUMS OF POWERS $i = \frac{(n+1)}{2}$ can be written as $\sum_{i=1}^n i$. So $i^2 = \frac{(n+1)(2n+1)}{6} = \sum_{i=1}^n i^2$ and so $i^3 = m \left[\frac{(n+1)}{2} \right]^2$. We can thus write the RULES for SIGMA NOTATION as:

$$\sum_{i=1}^n c = n[c], \quad \sum_{i=1}^n c[a_i] = c \sum_{i=1}^n [a_i], \quad \sum_{i=1}^n [a_i] + [b_i] = \sum_{i=1}^n [a_i] + \sum_{i=1}^n [b_i], \quad \text{and} \quad \sum_{i=1}^n [a_i] - [b_i] = \sum_{i=1}^n [a_i] - \sum_{i=1}^n [b_i].$$

Let us then express $\int_{[1]}^{[3]} [e^x] \partial[x]$ as a LIMIT OF SUMS. First, note that $\Delta[x] = \left(\frac{b-a}{n} \right) = (3 - 1)/n = \frac{2}{n}$. The value $\frac{2}{n}$ is the WIDTH of the INTERVAL. Thus, $x_0 = 1 + \frac{2}{n}$, $x_1 = 1 + \frac{4}{n}$, $x_2 = 1 + \frac{6}{n}$, $x_i = 1 + \frac{2i}{n}$. These are the intervals of the ENDPPOINTS.

Now observe that at the n^{th} term of the SEQUENCE $x_i = 1 + \frac{2i}{n}$, the index i is repeated.

The INTEGRAL can be represented as limits of sums of functions. So we write $\int_{[a]}^{[b]} [f(x)] \partial[x] = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta(x)$.

The symbol $\int$ (an elongated S) is the Integral sign, $f(x)] \partial[x]$ is the Integrand, while $[a]$ and $[b]$ are the upper and lower LIMITS OF INTEGRATION.

So $\int_{[a]}^{[b]} [f(x)] \partial[x]$ is a one symbol value and so are $\int_{[a]}^{[b]} [f(t)] \partial[t]$, $\int_{[a]}^{[b]} [f(r)] \partial[r]$ and, so forth. Note that when $\Delta[x] = \left(\frac{a-b}{n} \right)$, then the $\int_{[b]}^{[a]} [f(x)] \partial[x] = -\int_{[a]}^{[b]} [f(x)] \partial[x]$.

To understand AREA, we use the limit of a continuous function. The objective is to show that the SIGMA NOTATION distinctly defines R_n and L_n by SUMMING OVER REPEATED INDICES. This is decribed by the SUMMATION CONVENTION where we START at $i = 1$, and END at $[n]$. Thus the notation $\sum_{i=1}^n f(x_i)$.

12. Approximating Areas

The TANGENT LINE is used approxiamte the AREA [A] under a graph. The $\lim_{n \rightarrow \infty} L_n = A = \lim_{n \rightarrow \infty} R_n$ and as (n) increases, R_n becomes a better approximation to the Area of S. Therefore, we DEFINE the AREA [A] to be the limit of the SUMS OF AREAS of the approaching RECTANGLES.

The general notation for a limit then becomes $A = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} L_n = \frac{1}{n}$. We then ANALYZE the points on the Area of the Region [S].

Given $\frac{1}{n}$ strips on the interval [a,b] :

1. $x_0 = a$ and $x_n = b$
2. $\Delta[x] = \left(\frac{b-a}{n}\right)$
3. $(x^*_1, x^*_2, x^*_3, \dots \dots \dots, x^*_n)$ are the MIDPOINTS/SAMPLE POINTS
4. $[(x_0, x_1), (x_1, x_2), (x_2, x_3), \dots \dots \dots x_{n-1}, x_n)]$ are the SUBINTERVALS.

So the ENDPOINTS of the SUBINTERVALS $x_1 = a + \Delta[x]$, $x_2 = a + 2 \Delta[x]$, $x_3 = a + 3 \Delta[x]$, $\dots \dots \dots$,

Now that we have the points, the sum of the areas of the approaching rectangles are given by:

$$\sum_{i=1}^n f(x_i) \Delta(x) = \lim_{n \rightarrow \infty} f(x_1) \Delta(x) + f(x_2) \Delta(x) + \dots \dots \dots + f(x_n) \Delta(x) \text{ for } R_n$$

$$\sum_{i=1}^n f(x_{i-1}) \Delta(x) = \lim_{n \rightarrow \infty} f(x_0) \Delta(x) + f(x_1) \Delta(x) + f(x_2) \Delta(x) \dots \dots \dots + f(x_{n-1}) \Delta(x) \text{ for } L_n \text{ and}$$

$$\sum_{i=1}^n f(x^*_{i-1}) \Delta(x) = \lim_{n \rightarrow \infty} f(x^*_1) \Delta(x) + f(x^*_2) \Delta(x) + f(x^*_3) \Delta(x) \dots \dots \dots + f(x^*_n) \text{ for } M_n.$$

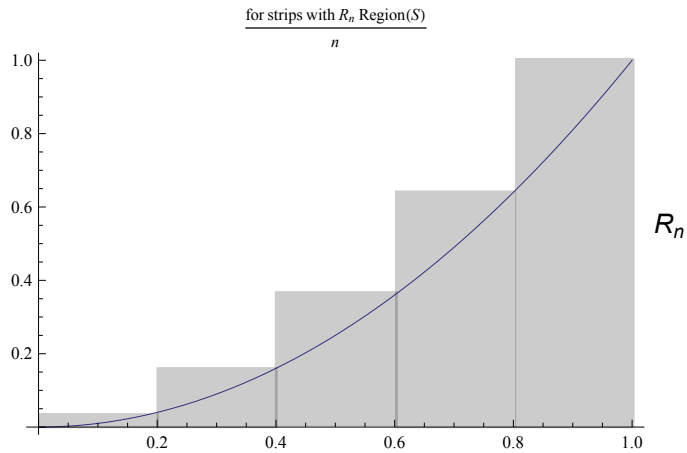
To show that the sum of the areas of the approaching rectangle is $\frac{1}{n}$, we note that the (VALUE) of the Area of n^{th} strip S = the (FUNCTION)

$$\Delta(x) * f(x_i) = f(x_i) \Delta(x).$$

13. Area of A Curve

Given a curve of the line $y = x^2$, let us now show that the sum of areas of the approaching rectangles is $\frac{1}{3}$. The graph of the curve is called a PARABOLA.

Plot $[x^2, \{x, 0, 1\}, \text{PlotLabel} \rightarrow R_n \text{ for Region } [S] \text{ with } \frac{1}{n} \text{ strips}]$



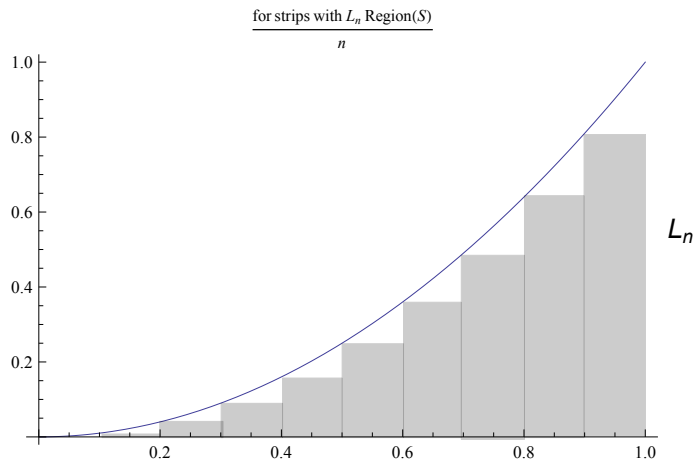
The Area of the Curve is approximated by an equation of the $\lim_{n \rightarrow \infty} R_n = \frac{1}{3} = S$. We note that the WIDTH of the RECTANGLE is $\frac{1}{n}$ and the HEIGHT is $f(x) = x^2 = y$ at the points $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n}{n}$. If we substitute for $\frac{1}{n}$ in the function, we obtain the points $(\frac{1}{n})^2, (\frac{2}{n})^2, (\frac{3}{n})^2, \dots, (\frac{n}{n})^2$.

So $R_n = (\frac{1}{n})^2 \frac{1}{n} + (\frac{2}{n})^2 \frac{1}{n} + (\frac{3}{n})^2 \frac{1}{n} + \dots + (\frac{n}{n})^2 \frac{1}{n}$ which is the sum of $f(x) \Delta(x)$.

We now must use the FORMULA for SUMS OF SQUARES for the first $[n]$ positive integers. We have $\frac{n(n+1)(2n+1)}{6} = 1^2 + 2^2 + 3^2 + \dots + n^2$. So $R_n =$

$\frac{1}{n^3} [1^2 + 2^2 + 3^2 + \dots + n^2]$. So substitute to obtain $R_n = \frac{1}{n^3} \left[\frac{n(n+1)(2n+1)}{6} \right] = \frac{1}{n^2} \left[\frac{(n+1)(2n+1)}{6} \right] = \frac{(n+1)(2n+1)}{6n^2} = \frac{1}{6} \left[\left(\frac{n+1}{n} \right) \left(\frac{2n+1}{n} \right) \right] = \frac{1}{6} [1 \cdot 2] = \frac{1}{3} = S$.

`Plot[x^2, {x, 0, 1}, PlotLabel -> Ln for Region [S] with  $\frac{1}{n}$  strips]`



From the the graphs, observe the following:

1. The sum of the Area of the Left-End-Points $L_n < \text{Area [A] of S}$; and
2. The sum of the Area of the Right-End-Points $R_n > \text{Area [A] of S}$.

We can see that the Area of [A] of S lies between the values of L_n and R_n . That is, $L_n \leq A \leq R_n$. So as the NUMBER OF STRIPS is INCREASED, both L_n and R_n approach the value $\frac{1}{3}$.

We also note that the left-most rectangle in L_n , has collapsed, because its height is 0 [reduced one rectangle]. This implies the INTERVAL for L_n ends at $[x_{i-1}]$.

The table below shows the estimates of the $A \approx 0.333134$, obtained by averaging the

n [number of Strips]	L _n (Left - Hand	R _n (Right - Hand
	End - Points)	End - Points)
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3385000
1000	0.3326335	0.3336335

Clearly, $0.3326335 \leq A \leq 0.3336335$. Recall from Chapter 7, that the FORMULA $S_{2[n]} = \frac{1}{3} T_n \frac{2}{3} M_n$, gives a better APPROXIMATION for R_n .

- ★ In defining the TANGENT, we first approximate the slope of the TANGENT LINE by the SECANT LINE, then take the LIMIT/AVERAGE VALUE of the approximations.
- ★ In finding AREAS UNDER A CURVE, we approximate the region of S by RECTANGLES, then take the LIMIT/AVERAGE VALUES of the areas of the rectangles as we increase the number of rectangles.

This is the complete derivation of the LIMIT OF A SEQUENCE. Find the definition in chapter 5.

14. Normal Distributions

People with Intillegent Quotients [IQs]of over 200 are VERY RARE. We can safely say this by using members of FAMILIES OF FUNCTIONS in normal distributions. This is in part of MULTIVARIABLE CALCULUS where we can verify a sample of ONE STANDARD DEVIATION OF THE MEAN.

For instance, we verify that $\int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma)^2} \partial[x] = 1$ where σ is the STANDARD DEVIATION(the distribution of [x]), and $[\mu]$ is the MEAN(the long run average of the random variable [x]).

The MEDIAN for the PROBABILITY DENSITY FUNCTION can be written as $\int_{[m]}^{[\infty]} [f(x)] \partial[x] = \frac{1}{2}$, and the MEAN for the same can be written as $[\mu] = x \int_{[-\infty]}^{[\infty]} [f(x)] \partial[x]$. This is possible because PROBABILITIES are MEASURED in the SCALE of 0 to 1.

We can also write the EXPONENTIAL DENSITY FUNCTION which has the form $[f(t)] = 0 \leq 0$.
 $ce^{-c(t)} \geq 0$

So in general, for $P[a \leq x \leq b] = \int_{[a]}^{[b]} [f(x)] \partial[x]$, the *Integral for the probability density function* can be written as $\int_{[-\infty]}^{[\infty]} [f(x)] \partial[x] = 1$ for all normal distributions or RANDOM PHENOMENA.

For instance, the function $F = \frac{A}{\int_{[b]}^{[t]} [C(t)] \partial[t]}$ is the FORMULA for ESTIMATING CARDIAC OUTPUT, where A is the amount of DYE, and the INTEGRAL is the approximation of a probed concentration readings. The RATE OF blood FLOW into the AORTA, is the VOLUME of blood pumped by the heart PER UNIT TIME.

We can also estimate the FLUX/DISCHARGE as the VOLUME of blood that passes a cross-section per-unit time. So $F = \frac{\pi P R^4}{8 \eta l}$ is proportional to the fourth power of the RADIUS of blood vessels, where [P] is PRESSURE, l is LENGTH, and η is a constant.

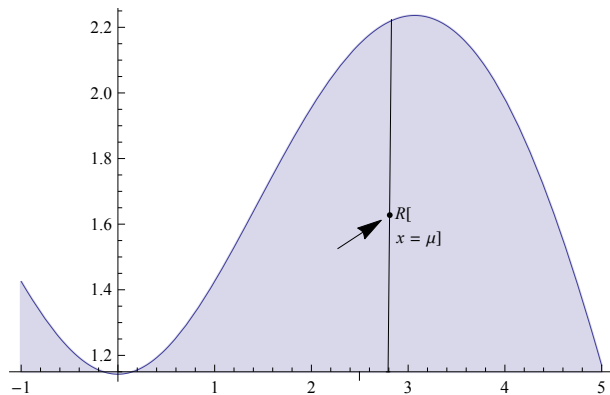
- ★ In the DYE DILUTION METHOD, dye is injected into the RIGHT ATRIUM and flows into the AORTA. A PROBE inserted into the aorta, MEASURES the CONCENTRATION of the dye leaving the heart at equally spaced times, over the INTERVAL $[0, t]$ until the dye has cleared. If we let $C[t]$ be the concentration of the dye at time $[t]$ and divide $[0, t]$ into SUBINTERVALS of equal length $\Delta(t)$, then the amount of dye that flows past the measuring point during the subinterval from $t = t_{i-1}$ to $t = t_i$ is APPROXIMATELY the (CONCENTRATION)*(VOLUME) = $C(t) [F\Delta(t)]$ where F is the RATE OF FLOW we're trying to DETERMINE.

Thus, the TOTAL AMOUNT OF DYE is approximately $\sum_{i=1}^n C(t) F\Delta(t) = F \sum_{i=1}^n C(t) \Delta(t)$ and letting $n \rightarrow \infty$, we obtain the total amount of dye $A = F \int_{[0]}^{[t]} C(t) \partial[t]$. Solving for F, we have the formula for determining the rate of flow/cardiac output.

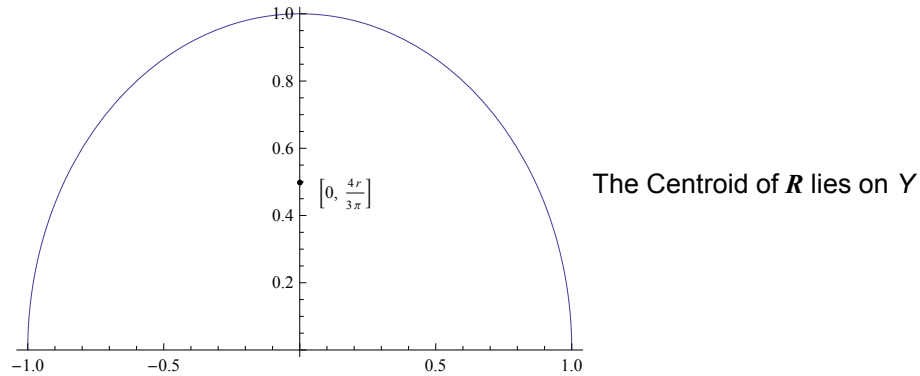
15. The Symmetry Principle

Suppose we have a FLAT PLATE with uniform density $[\rho]$ that occupies the region R . We can say that if R is SYMMETRIC about the line ll , then the **CENTROID of R lies on the line l** . The Centroid is the CENTER OF MASS of the semi-circular plate with radius $[r]$, and is located at the points $[X,Y]$. We can then define the SYMMETRY PRINCIPLE using the points $X = \frac{1}{A} \int_a^b x f(x) \partial[x]$ and $Y = \frac{1}{A} \int_a^b \frac{1}{2} [f(x)]^2 \partial[x]$. where X is the x-coordinate of the **CENTROID of R** . The expression for X resembles the MEAN. So **R balances at a point on the line $x = \mu$** .

```
Plot[-Cos[Sqrt[x^2/3 + 1]] + 2 Sin[Sqrt[x^2/3 + 1]], {x, -1, 5}, Filling -> Axis]
```



```
Plot[Sqrt[1 - x^2], {x, -1, 1}]
```

★ Note that the Centroid can be revolved to obtain SOLIDS OF REVOLUTION.

In addition, we can also say the FORCE required to maintain a SPRING stretched $[x]$ units beyond its natural length is proportional to $[x]$. So in this case $f(x) = K[x]$, where $K[x]$ is the SPRING CONSTANT. This is correct provided that $[x]$ is not too large.

Note that if f is continuous on the interval $[a, b]$, and $[n]$ is large, then $[\Delta x]$ is small. That is, the value of f does not change very much on the given interval $[x_{i-1}, x_i]$. So f is almost constant.

Therefore we can say that W , from $[x_{i-1}, x_i]$ is approximately given by the equation:

$W = f(x_i^*) \Delta(x) = \int_{[a]}^{[b]} f(x) \partial[x] = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta(x) = \text{Work Done}$. Similarly $F = M \frac{\partial^2 [t]}{\partial [t^2]}$, $W = F d$, and $F = M g$ where F , M , W , $[d]$ and $[g]$ represents FORCE, MASS, WORK DONE, DISTANCE and GRAVITY. Observe that $\frac{\partial^2 [t]}{\partial [t^2]} = a [\text{ACCELERATION}]$.

At this point, we can define the AVERAGE VALUE OF A FUNCTION as $f_{\text{ave}} = \frac{1}{b-a} \int_{[a]}^{[b]} f(x) \partial[x]$ on the interval $[a, b]$. Thus, the MEAN VALUE OF A FUNCTION $f(c) = f_{\text{ave}} = \frac{1}{b-a} \int_{[a]}^{[b]} f(x) \partial[x]$. we can rearrange the equation to obtain $f(c) (b-a) = \int_{[a]}^{[b]} f(x) \partial[x]$.

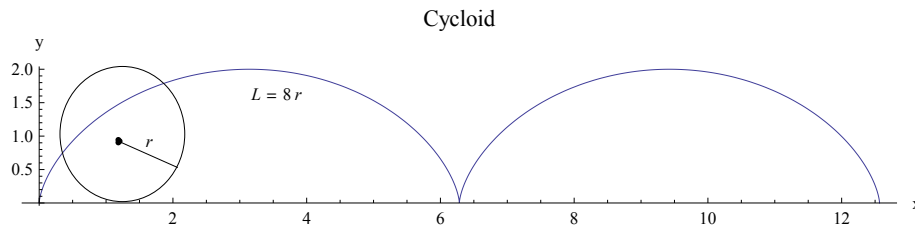
For instance, let $f(x) = 1 + x^2$ on the interval $[-1, 2]$. The MEAN VALUE implies that there is a number $[c]$, on $[-1, 2]$ such that $\int_{[a]}^{[b]} f(x) \partial[x] = f(c) (b-a)$. We then solve $f(x) = 1 + x^2$ using the the integral $\int_{[-1]}^{[2]} (1 + x^2) \partial[x]$. We then substitute to obtain $\int_{[-1]}^{[2]} \left(x + \frac{x^3}{3}\right)_{x=[-1, 2]} = 2$, if $[x] = 1$. So, $2 = f(c) (b-a) = f(c) (2 - (-1)) = f(c)$.

Since $2 = f(c)$, we can solve explicitly for (c) by forming the equation $f(c) = 1 + c^2$. Substituting, $2 = 1 + c^2$, we obtain $c^2 = 1$ and so $f(c) = f_{\text{ave}}$.

16. The Cycloid

Let us evaluate the statement that one ARC LENGTH of the CYCLOID is 8 times the RADIUS of the generating CIRCLE.

ParametricPlot[{x-Sin(x),1-Cos(x)},{x,0,4π},AxesLabel→{"x","y"},PlotLabel→"Cycloid",AspectRatio→Automatic]



If $x = f(t)$, and $y = g(t)$ for $a \leq t \leq b$ and is TRAVERSED exactly once as (t) increases from (a) to (b) , then its length

$$L = \int_{[a]}^{[b]} \sqrt{\left(\frac{\partial(x)}{\partial t}\right)^2 + \left(\frac{\partial(y)}{\partial t}\right)^2}.$$

Since $\frac{\partial(x)}{\partial \theta} = r(1 - \cos \theta)$ and $\frac{\partial(y)}{\partial \theta} = r(\sin \theta)$, we substitute to obtain $L = \int_{[a]}^{[b]} \sqrt{r^2(1 - \cos \theta)^2 + r^2(\sin^2 \theta)} \, d\theta = \int_{[0]}^{[2\pi]} \sqrt{r^2(1 - \cos \theta)^2 + r^2(\sin^2 \theta)} \, d\theta$ for $0 \leq \theta \leq 2\pi$.

We then expand the equation to obtain $\int_{[0]}^{[2\pi]} \sqrt{r^2(1 - 2\cos \theta + \cos^2 \theta + \sin^2 \theta)} \, d\theta = r \int_{[0]}^{[2\pi]} \sqrt{2(1 - \cos \theta)} \, d\theta = L$. This result can be verified when we calculate the AREA [A] of the ROLLING CIRCLE/ ONE ARCH OF THE CYCLOID.

To prove the result, Let $A = \int_{[a]}^{[b]} y \, dx$. We use the PRODUCT RULE from chapter 9 to rewrite the

equation as $A = \int_{[a]}^{[b]} g(x) f'(x) \, dx$. If $x = f(t)$, and $y = g(t)$ for $\alpha \leq t \leq \beta$. Then $A = \int_{[\alpha]}^{[\beta]} g(t) f'(t) \, dt$. Thus, we can interpret the area of one arch of the cycloid to be $0 \leq \theta \leq 2\pi$ for $x = r(\theta - \sin \theta)$ and $(y) = r(1 - \cos \theta)$. We then substitute to obtain

$$A = \int_{[a]}^{[b]} y \, dx = \int_{[a]}^{[b]} g(x) f'(x) \, dx = \int_{[a]}^{[b]} r(1 - \cos\theta) r(1 - \cos\theta) \, d\theta = r^2 \int_{[0]}^{[2\pi]} (1 - \cos\theta - \cos\theta + \cos^2\theta) \, d\theta =$$

$$r^2 \int_{[0]}^{[2\pi]} (1 - 2\cos\theta + \cos^2\theta) \, d\theta = r^2 \int_{[0]}^{[2\pi]} \left[(1 - 2\cos\theta) + \frac{1}{2}(1 + 2\cos\theta) \right] \, d\theta = r \int_{[0]}^{[2\pi]} \sqrt{2(1 - \cos\theta)} \, d\theta = 8r = A = L$$

★ Note that the DIAMETER of the ROLLING CIRCLE is $2\pi r$.

★ $\int_0^{2\pi} \sqrt{2(1 - \cos(x))} \, dx = 8$

17. Estimating The Volume Of Parametric Curves

The function $A(x) = \pi y^2 = \pi(r^2 - x^2)$ when $y = \sqrt{r^2 - x^2}$, is the RADIUS (r) in the plane $P(x)$ that intersects the sphere in a circle. Thus, the $\int_a^b A(x) \, dx$, $P(x) = A(x) = \pi(\text{radius})^2$ forms a DISK in the sphere.

For an ANNULAR RING, $A = \pi(r_{\text{out}})^2 - \pi(r_{\text{in}})^2$ where r_{out} is the OUTER RADIUS and r_{in} is the INNER RADIUS of the RING.

The PYTHAGOREAN RELATION $x^2 + y^2 = r^2$ is applied to obtain the AREA OF PARAMETRIC CURVES, and thus $y^2 = r^2 - x^2$.

So to ESTIMATE THE VOLUME OF PARAMETRIC CURVES, we also use the $\int_a^b A(x) \, dx$ where $A(x) = \pi y^2 = \pi(r^2 - x^2)$

18. Systems of Differential Equations

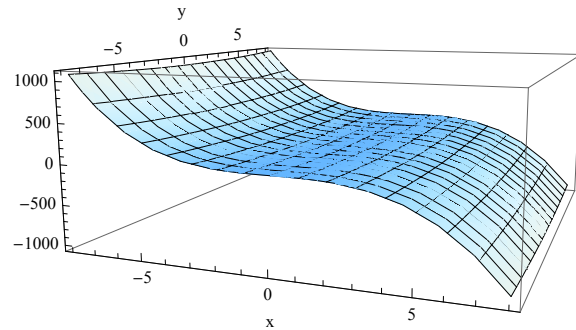
Differential Equations are used to MODEL PHYSICAL PHENOMENA e.g the RELATIVE GROWTH RATE of a population. Given the INITIAL CONDITIONS, we can describe LAWS that occur frequently in nature.

For instance, let the $\int \frac{dy}{y} = \int K dt$. We can determine the INITIAL VALUE (A) of this function by rearranging to obtain $\int \frac{dy}{dt} = \int Ky$. So when $y(0) = y_0$, $y(t) = y_0 e^{Kt}$. We've obtained the function $y(t) = y_0 e^{Kt}$ by solving $|y| = Kt + C$ which is equivalent to $|y| = e^{Kt+C} = e^C e^{Kt}$. Thus $y = (A) e^{Kt}$ where $(A) = \pm e^C$ or 0 [the initial value] So we've derived the function of the LAW OF NATURAL DECAY represented by $y(t) = y_0 e^{Kt}$.

In LINEAR APPROXIMATION, the function $L(x) = x + 1$ can be interpreted as starting out along the TANGENT LINE and stopping at the horizontal distance traveled (x); called the STEP SIZE. For instance, let $y' = F(x, y)$ where $y(x_0) = y_0$. We can then approximate the values of the function at equally spaced numbers using the general notation $y_n = y_{n-1} + hF[x_{n-1}, y_{n-1}]$ where (h) is the STEP SIZE and the equally spaced numbers take the form $x_0, x_1 = (x_0 + h), x_2 = (x_1 + h), \dots, x_n = (x_{n-1} + h)$.

- ★ If $\frac{dy}{dx} = g(x) f(y)$, then $\frac{dy}{dx} = \frac{g(x)}{h(y)}$ for $f(y) \neq 0$. Note that $h(y) = \frac{1}{f(y)}$. So the function can be written in differential form as $h(y) dy = g(x) dx$ and integrated on both sides to obtain $\int h(y) dy = \int g(x) dx$. We have defined (y) IMPLICITLY/SEPERABLE EQUATION as function of (x). So given the DIFFERENTIAL EQUATION $\frac{dy}{dx} = \frac{6(x^2)}{2(y) + \cos(y)}$, we can solve the function by $\int 2(y) + \cos(y) = \int 6(x^2) dx$. Thus $y^2 + \sin(y) = 2(x^3) + C$. So far this is GENERAL SOLUTION because we cannot express (y) as function of (x). But given the INITIAL CONDITION $y(1) = \pi$ and $(x) = 1$, we substitute on the general solution to obtain $\pi^2 + \sin \pi = 2 + C$. Thus $C = \pi^2 - 2$. Finally, we substitute for (C) and the SOLUTION of $\frac{dy}{dx} = \frac{6(x^2)}{2(y) + \cos(y)}$ can then be written explicitly as $y^2 + \sin(y) = 2(x^3) + \pi^2 - 2$ and GRAPHED.

`Plot3D[y^2 + Sin[y] == 2 x^3 + pi^2 - 2, {x, -8, 8}, {y, -8, 8}, AxesLabel -> {"x", "y"}]`



★ The GRAPHIC and NUMERICAL APPROACH, are MATHEMATICAL MODELS.

19. A Note For Casual Students

Learning Mathematical skills is the ability to solve Mathematical problems arising in practice. But what can we do with these learning? Calculus is primarily a TOOL. You ought to decide then, whether you should stress on its applications, by appeal to intuition, or rather by extensive drill of problems to develop manipulative skills. This text is the former.

It is often a challenge to explain matters thoroughly. However, it is understood that DIFFERENTIATION is the process of deriving RATES OF CHANGE. In this case, we ought to see, that EQUATIONS do not always have solutions. That FUNCTIONS and their DERIVATIVES may not exist at every point leading to problems in GRAPH SKETCHING and OPTIMIZATION. So, learning feeble Mathematical skills is soft intellectual trash. We therefore proceed in the following chapters to define important of all, the ELEMENTARY RULES for calculations.

SECTION II: ARITHMETIC

20. Complex Numbers

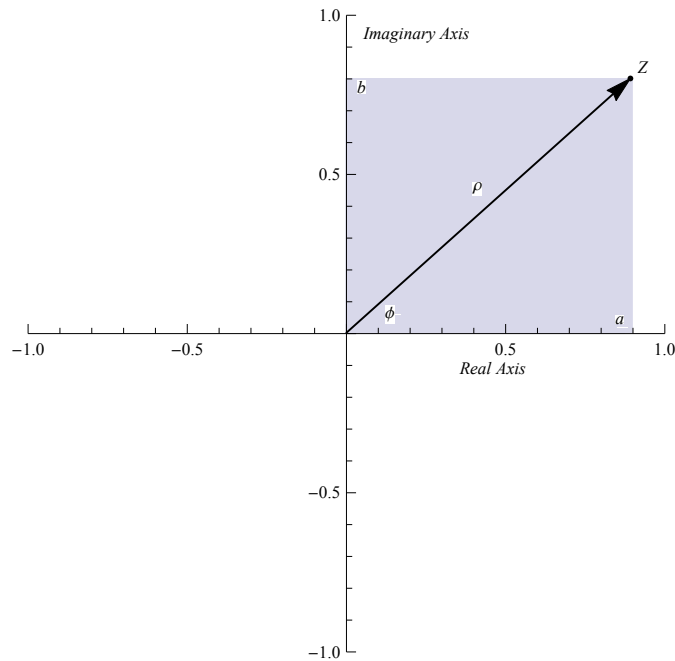
For certain calculations, we can determine the roots of an algebraic equation using the general numerical procedures for non-linear equations. I'll let the reader decide whether most of these methods adhere to the principle of Mathematics as outlined in the preface.

We know that every equation of degree (n), whose coefficients are REAL, or COMPLEX NUMBERS, has (n) real or complex ROOTS. It is also known to be difficult to give a formula, or a finite sequence of formulae which produce the roots of an equation of degree 5 or higher.

The (n^{th}) power of a complex number can be evaluated using the trigonometric form $[\rho(\text{Cos}\phi + i\text{Sin}\phi)]^n$. In particular, we have $i^2 = -1$ ($\sqrt{-1} = i$), $i^3 = -i$, $i^4 = +1$ and in general, $i^{4n+k} = i^k$. So the expression $\rho^n[\text{Cos}(n)\phi + i\text{Sin}(n)\phi]$ implies that the MAGNITUDE ρ , is raised to the n^{th} power, and the argument ϕ is multiplied by (n).

For instance, CONJUGATE COMPLEX NUMBERS have the same MAGNITUDE. This implies, the REAL PARTS are equal, and the IMAGINARY parts only differ in the SIGN.

Let $Z = a + bi = \rho(\text{Cos}\phi + i\text{Sin}\phi) = \rho e^{i\phi}$, then $Z^* = a - bi = \rho(\text{Cos}\phi - i\text{Sin}\phi) = \rho e^{-i\phi}$. So, $\text{Re}(Z^*) = \text{Re}(Z)$ and $\text{Im}(Z^*) = -\text{Im}(Z)$.



Complex numbers can be represented in three forms:

1. Algebraic, where $Z = 1 + i\sqrt{3}$
2. Trigonometric, where $Z = 2\left[\cos \frac{\pi}{3} + i\sin \frac{\pi}{3}\right]$ and
3. Exponential, where $Z = e^{i\pi/3}$.

The expression $Z = \rho e^{i\phi}$ where ρ is the MAGNITUDE and ϕ is the ARGUMENT, can be rewritten using the FORMULA $e^{i\phi} = \cos\phi + i\sin\phi$. At this point, we can define the LENGTH of the POSITION/RADIUS VECTOR as $\rho = |Z|$ where $|Z|$ is the MAGNITUDE of the COMPLEX NUMBER.

Thus, the ANGLE ϕ , given in RADIAN MEASURE, is the ARGUMENT of the complex number, and is denoted by $\arg Z$, which is the PRINCIPLE VALUE of the COMPLEX NUMBER

Let us then relate the trigonometric form of the complex number $Z = \rho[\cos\phi + i\sin\phi]$ to the POLAR COORDINATES.

First, note that the ALGEBRAIC FORM of the COMPLEX NUMBER is in the form $(Z = a + bi)$ where $a = \operatorname{Re}(Z)$ and $b = \operatorname{Im}(Z)$. So for $a = 0$, $Z = bi$ which is a

PURE IMAGINARY NUMBER.

★ Note that every COMPLEX NUMBER corresponds to a POSITION/RADIUS VECTOR.

21. Algebraic Expressions

First, it is understood that the BINOMIAL INEQUALITY is represented by the $|a - b| \leq \frac{1}{2}[a^2 + b^2]$.

Lets us analyze the SPECIAL LOGARITHMS i.e. $\lg 324 = 2.5105$. We know that the DECIMAL LOGARITHM is written in the form $\log_{10}(x) = \lg x$ and the $\log(x 10^\alpha) = \alpha + \log x$.

The BINARY LOGARITHM is written in the form $\log_2(x) = \lg x$.

So $\log x = K + \log \hat{x}$, we evaluate K as the CHARACTERISTIC, and $\log \hat{x}$ as the sequence behind the decimal, called the MANTISSA (mostly found in the LOGARITHM TABLES). For instance, 324000, 3240, 324, 0.0324, have the same mantissa 5105, but different characteristics.

The $\log 3 + 2 \log x + \frac{1}{3} \log y - \log 2 - \log x - 3 \log u = \frac{3x^2 3\sqrt{y}}{2zu^3}$ implies the logarithm of an EXPRESSION.

The $\log_a(x) = M \log_b(x)$ where $M = \log_a(b) = \frac{1}{\log_b(a)}$ is the MODULUS OF TRANSFORMATION M .

The expressions above are used to calculate the logarithm of PRODUCTS and FRACTIONS, as SUMS or DIFFERENCES of logarithms. We can then say that the *determination of a number, or an expression, from its logarithm* is the process of RAISING TO A POWER. That is, $b^u = x$ yields $\log_b(x) = u$ where $x > 0$, base $b > 0$, and $b \neq 1$.

Thus, the $\log x^n = n \log x$, $\log(xy) = \log x + \log y$, $\log \frac{x}{y} = \log x - \log y$, and in particular, $\log \sqrt[n]{x} = \frac{1}{n} \log x$.

$a^{\frac{p}{q}} = q \sqrt[q]{a^p}$ implies the q^{th} root of (a) to the power (p) . So $a^x = e^{x \ln a}$ implies $\ln a$ is the natural logarithm of (a) where $e = 2.718281828459$.

$a_1 + a_2 +, \dots, a_n = \sum_{k=1}^n a_k$ implies that the SUMMATION SIGN (Σ), is used to denote the sum of (n) SUMMANDS $a_k (k = 1, 2, \dots, n)$ where k is the RUNNING INDEX/SUMMATION VARIABLE.

$a_1, a_2, \dots, a_n = \prod_{k=1}^n a_k$ implies that the PRODUCT SIGN ($\prod$), is used to denote the product of (n) FACTORS $a_k (k = 1, 2, \dots, n)$ where k is the RUNNING INDEX. For instance, $\prod_{i=1}^6 i^2 = 518400$.

22. Methods Of Proof

When we write that the existence of a 3rd Degree interpolation spline function results in a tridiagonal linear equation system, which has a unique solution, we're describing the steps of a PROOF, that gives a method of calculation for a result[SOLUTION] which satisfies the PROPOSITION(s). This is called a CONSTRUCTIVE PROOF.

When we write $(p \rightarrow q)$ for a NATURAL NUMBER for n_0 , $n \geq n_0$, it follows that $(n + 1, n \geq n_0)$ is a statement of logical deduction. This is called MATHEMATICAL INDUCTION.

When we write $(\bar{q} \rightarrow q)$ stating a false assumption that can only result in a false conclusion, if $(\bar{q})$ is FALSE, then q must be TRUE, is a proof by CONTRADICTION, to prove the statement the q .

- ★ When we write $[p \leftrightarrow q]$ as an equivalence, the ARITHMETIC OPERATION must be uniquely invertible. That is, if $0 < a^{n+1} < 1$, then $1 - a^{n+1} < 1$ is a DIRECT PROOF.
-

23. Geometric Representation of Numbers

If $a = m c$, and $b = n c$ where $[m, c] \in \mathbb{Z}$, it follows that $\frac{a}{b} = \frac{m}{n} = x$ which is **RATIONAL**. Here (a) and (b) are commensurable, if both are an INTEGER multiple of a third number (c) . Thus, we define **COMMENSURABILITY** as a NUMBER measurable by the same number. Is that all?

The connected SET of REAL numbers with endpoints (a) and (b) where $a < b$, and (a) is allowed to be the INTERVAL $-\infty$, and (b) is allowed to be $(+\infty)$, is the INTERVAL OF NUMBERS. We write that (a, b) , an OPEN INTERVAL, by using brackets, and $[a, b]$ as a CLOSED INTERVAL by using parenthesis.

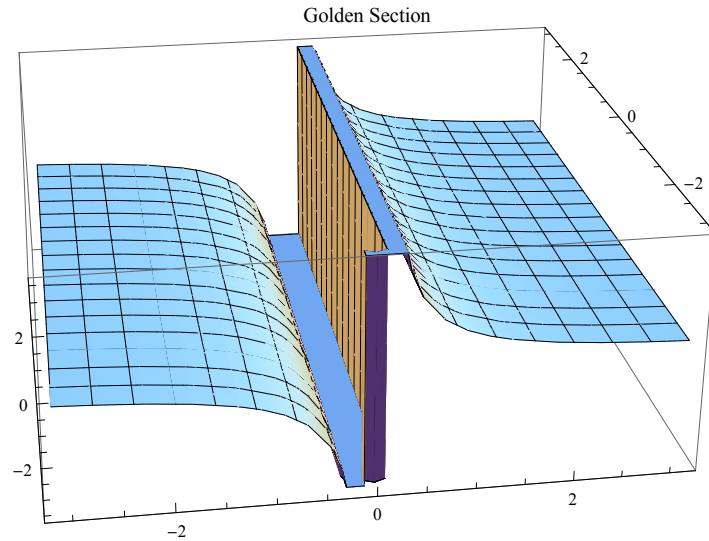
So $(a, b]$ or $[a, b)$ represent HALF-OPEN or HALF-CLOSED INTERVALS.

ARITHMETIC OPERATIONS can be performed with any two REAL NUMBERS, and the result is real number although

$$ZZ^* = \rho^2 = |Z|^2 = a^2 + b^2 = (a + bi)(a - bi).$$

The set of real numbers can then be defined as RATIONAL and IRRATIONAL numbers with FINITE, ORDERED, DENSE EVERYWHERE and CLOSED properties. For instance, the solution of the GOLDEN SECTION $\frac{x}{a} = \frac{(x-a)}{x}$, leads to the QUADRATIC EQUATION $x^2 + x - 1 = 0$ if $(a) = 1$.

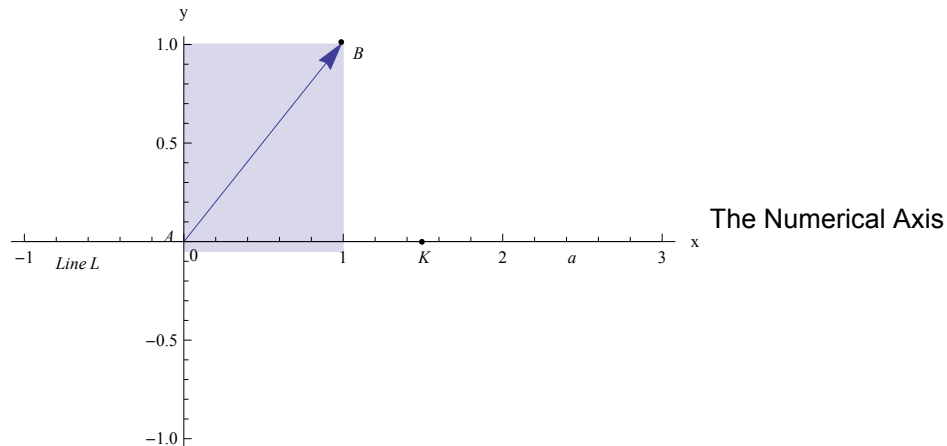
Plot3D[x/1==x-1/x,{x,-Pi,Pi},{y,-Pi,Pi},PlotLabel->"Golden Section"]



The solution $x = \frac{(\sqrt{5}-1)}{2}$ is a QUADRATIC IRRATIONAL. It contains the IRRATIONAL NUMBER $\sqrt{5}$. Thus, we note that QUADRATIC IRRATIONALS are **square-free numbers**. They have the form $\frac{a+b\sqrt{D}}{c}$ where (a), (b), and (c) are INTEGERS, (c) $\neq 0$, $D > 0$ and is the square-free number.

The IRRATIONAL NUMBERS which are not **ALGEBRAIC IRRATIONALS** are the TRANSCENDENTAL NUMBERS (π) and (e).

We can thus say that the REAL ROOTS of the algebraic equation $a_n x^n + a_{n-1} x^{n-1} + a_1 x^1 + a_0 = 0$ belong to the IRRATIONAL NUMBERS when $n > 1$ are the INTEGER COEFFICIENTS.



If we rotate the diagonal of AB of the UNIT SQUARE, so that B goes into the POINT K , then K does not have a RATIONAL COORDINATE.

Every RATIONAL NUMBER corresponds to a certain POINT on the line (L). If we fix an ORIGIN(the ZERO POINT), a positive direction(orientation), and a UNIT LENGTH (L) as the MEASURING RULE, then the point (a) is a RATIONAL POINT, and the line (L) is the NUMERICAL AXIS.

Thus, if (a) $\neq 0$, then no rational number (b) exists such that ($b \cdot 0$) = a could be fulfilled. This implies that DIVISION BY ZERO IS NOT POSSIBLE.

Moreover, between any two different rational numbers (a) and (b) where ($a < b$), there is atleast one rational number (c), where ($a < c < b$). This implies that the SET OF RATIONAL NUMBERS is DENSE EVERYWHERE,, and their ENUMERATION and ORDERING are the ELEMENTARY RULES FOR CALCULATION.

★ Note too, that every IRRATIONAL NUMBER can be represented as non-periodic infinite decimal fraction.

24. Inverse Derivatives

Let $x = f(u)$ and $y = g(u)$. Then $r = xi + yj = f(u)i + g(u)j$, joining the origin to the point $P(x, y)$ called the RADIUS/POSITION VECTOR.

If a , b , and c are VECTORS, then:

$a + b = b + a$; $a(b + c) = (a + b)c$; $k(a + b) = ka + kb$. Thus, i and j are called UNIT VECTORS and have a MAGNITUDE of 1 in the x and y positive axis.

SCALAR quantities have MAGNITUDE only i.e. time, temperature and speed. While VECTOR quantities have both MAGNITUDE and DIRECTION i.e. Velocity and Acceleration.

So, PARABOLAS, ELLIPSES, and HYPERBOLAS, make up the CLASS OF CURVES called the CONIC SECTION. They can be defined GEOMETRICALLY as the intersection of planes with the surface of the right circular cone.

In general, ASYMPTOTES of the HYPERBOLA $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are the lines $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$. Points on the hyperbola get closer and closer to the asymptotes, as they recede from the origin.

The graph of the equation $y = cx$ where (c) is a non-zero constant is called a PARABOLA, while $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ELLIPSE(bounded), and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a HYPERBOLA as any of the curves obtained from them by TRANSLATIONS and ROTATIONS.

If f' is positive on an interval, then f is INCREASING on that interval; and if f' is negative on an interval, the f is DECREASING on that interval.

Therefore, we can solve for a function $[y]$ in the equation $xy + x - 2y - 1 = 0$. This is because an equation $f[x, y] = 0$ defines $[y]$ IMPLICITLY as a function of $[x]$. The derivative of y' can be solved by IMPLICIT DIFFERENTIATION.

If $y = x^2$, then $x = f^{-1}(y) = \sqrt{y} = y^{\frac{1}{2}}$. So when $y = x^3$, the $x = f^{-1}(y) = \sqrt[3]{y} = y^{\frac{1}{3}}$. The derivative of $y = x^3$ is then $\frac{\partial x}{\partial y} = \frac{1}{3x^2} = \frac{1}{3(3\sqrt{y})^2} = \frac{1}{3} \left(y^{\frac{1}{3}}\right)^2 = \frac{1}{3(y)^{\frac{2}{3}}}$. Thus, the

result is the INVERSE DERIVATIVE of the function $f(x) = x^3$. We note that $x = f^{-1}(y)$ and $y \neq 0$. The same applies for trigonometric functions.

Let $f(x) = \sqrt{x}$. Since for $y = f(x)$, it follows that $y = \sqrt{x}$ and so $x = y^2$. Therefore, observe that $g(y) = y^2$. This is the inverse of the function f and is denoted f^{-1} .

For instance, where $x = f^{-1}(y)$, $\frac{\partial x}{\partial y} = 1 / \frac{\partial y}{\partial x}$. If we let $y = f(x) = x^2$ for $x > 0$, then $x = f^{-1}(y) = \sqrt{y}$. So $\frac{\partial y}{\partial x} x^2 = 2x$, $\frac{\partial x}{\partial y} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}$. Thus, $\frac{\partial}{\partial x} \sqrt{y} = \frac{1}{2\sqrt{y}}$.

We thus list the notations for the higher derivatives as follows:

$$y' = f' x = \frac{\partial y}{\partial x} = D_x y = 1^{\text{st}} \text{ Derivative}$$

$$y'' = f'' x = \frac{\partial^2 y}{\partial x^2} = D^2_x y = 2^{\text{nd}} \text{ Derivative}$$

$$y''' = f''' x = \frac{\partial^3 y}{\partial x^3} = D^3_x y = 3^{\text{rd}} \text{ Derivative, and}$$

$$y^n = f^n = \frac{\partial^n y}{\partial x^n} = D^n_x y = n^{\text{th}} \text{ Derivative}$$

25. The Slope Of A Line

We can find the ORIGIN of the equation $x^2 + y^2 - 4x - 10y + 20 = 0$ by COMPLETING THE SQUARE for any equation in the form $x^2 + y^2 + Ax + C = 0$.

The equation of the circle sometimes appears in disguised form. For example, $x^2 + y^2 + 8x - 16y + 21 = 0$ turns out to be equivalent to $(x + 4)^2 + (y - 3)^2 = 4$. So then this becomes the STANDARD EQUATION OF THE CIRCLE with CENTER $[-4, 3]$ and RADIUS 2. In general terms, in this case, we find the num-

ber that must be added to the sum $x^2 + Ax$ to obtain a SQUARE. So, $\left(x + \frac{A}{2}\right)^2 = x^2 + Ax + \left(\frac{A}{2}\right)^2$. Thus, $x^2 + 8x + \left(\frac{8}{2}\right)^2 = x^2 + 8x + 16 = (x + 4)^2$; and $\left(y + \frac{A}{2}\right)^2 = y^2 + Ay + \left(\frac{A}{2}\right)^2 = y^2 - 6y + \left(\frac{-6}{2}\right)^2 = y^2 - 6y + 9 = (y - 3)^2$. We MUST add $(16 + 9)$ to the right hand of the equation to obtain the result $(x^2 + 8x + 16) + (y^2 - 6y + 9 + 21) = (16 + 9) - 21 = 25 - 21 = 4$. Thus $(x + 4)^2 + (y - 3)^2 = 4$.

The standard equation of the circle with the center at the origin $[0, 0]$ and radius (r) is thus $x^2 + y^2 = r^2$. For example, $x^2 + y^2 = 1$ is the equation of the circle with center at the origin and radius 1. So the graph of $x^2 + y^2 = 5$ is the circle with center at the origin and radius $\sqrt{5}$.

The CIRCLE, is therefore said to be the GRAPH of the given equation. This implies the SET OF POINTS satisfying the equation. Thus, the graph of the equation $(x + 3)^2 + y^2 = 2$ is the circle with center $[-3, 0]$ and radius $\sqrt{2}$. We can then say that the circle with center $[2, -1]$ and radius 3 has the equation $(x - 2)^2 + (y + 1)^2 = 9$.

Now, for a point $P(x, y)$ to lie on the circle with center $C[a, b]$ and radius (r) , the DISTANCE $\overline{PC}$ must be equal to (r) . Since the DISTANCE FORMULA $\overline{PC} = \sqrt{(x - a)^2 + (y - b)^2}$, P lies on the circle if and only if $(x - a)^2 + (y - b)^2 = r^2$. This is the STANDARD EQUATION OF THE CIRCLE.

So to derive the the y – intercept and the slope intercept equation for 2, we use the POINT SLOPE EQUATION. If we let L be the line through points $[3, -1]$ and $[2, 3]$, we can find its slope M , and the two point slope equations of L .

This is so if we let the line L to be horizontal, then the slope of the line L is ZERO. If we let the line L to be vertical, the slope of the line L is UNDEFINED. Thus, the slope of the line L increases without bound from zero when L is horizontal, and to $+\infty$ when the line L is vertical. The argument is similar for a negative sloped line. If the slope of M of the line L is known, then each point $[x, y]$ of L yields a POINT SLOPE EQUATION of L . **Hence, there is an infinite number of point slope equations for L .** Thus, $y_2 - y_1 = m(x_2 - x_1)$. From this, we have *the steepness of a line defined by the slope M* where $M = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)$. This implies that *the SLOPE is the RATIO OF CHANGE in the y – coordinate* to the corresponding change in the x – coordinate.

Therefore, PROOFS by means of COORDINATES is called ANALYTIC; in contrast to so-called SYNTHETIC PROOFS by AXIOMS. Analytically, the segment joining the midpoints of two sides of a triangle is one-half the length of the third side. So $\overline{M_1 M_2} = \sqrt{\left(\frac{u}{2} - \frac{u+b}{2}\right)^2 + \left(\frac{v}{2} - \frac{v}{2}\right)^2} = \sqrt{\left(\frac{b}{2}\right)^2} = \frac{b}{2}$. We thus find the midpoint of the segment connecting $[-5, 1]$ and $[1, 4]$ by using the MIDPOINT FORMULA $x = \frac{x_1 + x_2}{2}$ and $y = \frac{y_1 + y_2}{2}$. This relation is useful when solving the INEQUALITY $5 < 3x + 10 \leq 16$ for then we're determining the INTERVAL OF NUMBERS whether they are OPEN, CLOSED, HALF-OPEN, or HALF-CLOSED.

So, to find the DISTANCE between points $[1, 4]$ and $[5, 2]$, we use the DISTANCE FORMULA $\overline{P_1 P_2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

26. Differentiation Rules

If $f(x) = x^2 + 3$ and $g(x) = 2x + 1$, then $f[g(x)] = (2x + 1)^2 + 3 = 4x^2 + 4x + 7$. So $\frac{\partial y}{\partial x}$ of $4x^2 + 4x + 7 = 8x + 4$. Thus we can write $\frac{\partial}{\partial x}$ of $f[g(x)] = 8x + 4$. We can use the an alternative method called the CHAIN RULE, to compute the DERIVATIVE OF A COMPOSITE FUNCTION as $\frac{\partial}{\partial x} f'[g(x)] * g'(x)$ so that the derivative of $[f * g]$ is the product of the derivative of the derivative of the function f evaluated at $g(x)$ and then (x) .

Let $u = g(x)$, and $y = f(u)$. Then the composite function of f and g is $y = f[g(x)]$. We use the CHAIN RULE FORMULA $\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial x}$ which is equivalently

$\frac{f u}{\partial x} = \frac{f u}{g x} \frac{g x}{\partial x}$. For instance, let $y = u^3$ and $u = 4x^2 - 2x + 5$. Then the composite function $y = (4x^2 - 2x + 5)^3$ has the derivative $y' = \frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial x} = 3u^2(8x - 2)$. If we substitute for u , we obtain $y' = 3(4x^2 - 2x + 5)^2(8x - 2)$.

In general, the COMPOSITE FUNCTION $f[g]$ is defined as $f[g] = f[g(x)]$. The function (g) is applied first, then f . Thus, $g(x)$ is the INNER FUNCTION, and $f[g(x)]$ is the OUTER FUNCTION. In other words, $f[g]$ is called the COMPOSITION of g and f .

To make the concept stick, let $f(x) = x^2$ and $g(x) = x + 1$. Then $f[g] = f[g(x)] = (x + 1)^2 = x^2 + x + 1$ and $g[f] = g[f(x)] = x^2 + 1$. In this case, $f[g] \neq g[f]$. When f and g are differentiable, then so is their composition $f[g]$.

Observe that $f[g]$ is a **Binomial Composition**. Thus, we can write that EVERY POLYNOMIAL IS DIFFERENTIABLE, and its derivative can be computed using the arithmetic operation of the SUM, POWER, IDENTITY, and CONSTANT FUNCTIONS. For instance, let $f(x) = x^3 + 7x + 5$. Then $f'(x) = 3x^2 + 7 = 3(x^{n-1}) + \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(C) = 3x^2 + 7 + 0$. So now, we can define the DIFFERENTIATION RULES for FUNCTIONS as follows:

1. The derivative of a CONTANT, $\frac{\partial}{\partial x}(C) = 0$
2. The derivative of an IDENTITY, $\frac{\partial}{\partial x}(x) = 1$
3. The DIFFERENCE RULE where $\frac{\partial}{\partial x}(Cu) = C \frac{\partial u}{\partial x}$
4. The SUM RULE where $\frac{\partial}{\partial x}(u + v + \dots) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \dots$
5. The PRODUCT RULE where $\frac{\partial}{\partial x}(uv) = u \frac{\partial v}{\partial x} + v \frac{\partial u}{\partial x}$
6. The QUOTIENT RULE where $\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{v \frac{\partial u}{\partial x} - u \frac{\partial v}{\partial x}}{v^2}$ provided that $v \neq 0$
7. $\frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ provided that $x \neq 0$ and
8. The POWER RULE where $\frac{\partial}{\partial x}(x^m) = mx^{m-1}$

Note that the Rule 7 and Rule 8 are equivalent if $m = -1$.

So far, in finding $f'(x)$ when $f(x) = |x|$, we first note that the function is CONTINUOUS for all values of (x) . This implies, for $x < 0$, $f(x) = -x$. So, the process of finding the derivative of a function is DIFFERENTIATION at EVERY REAL NUMBER.

Thus, a function is said to be DIFFERENTIABLE, at a point x_0 , if the derivative of the function EXISTS at that point. Then DIFFERENTIABILITY IMPLIES CONTINUITY (but the converse is false).

If we let x_0 be any number in the DOMAIN of the function $y = f(x)$, let Δx denote a change from (x_0) to $(x_0 + \Delta x)$, and let Δy denote the corresponding

change in (y), then the RATIO $\frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{\text{Change in } (y)}{\text{Change in } (x)} = \frac{\Delta f(x)}{\Delta x} = \text{AVERAGE RATE OF CHANGE}$.

The derivative of a function is then denoted by any of the following EXPRESSIONS:

$$D_x y = \frac{dy}{dx} = y' = f'(x) = \frac{d}{dx} y = \frac{d}{dx} f(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \Big|_{x=0}.$$

Therefore, the DERIVATIVE, of f at x_0 , is thus defined as the INSTANTANEOUS RATE OF CHANGE of f at x_0 , which implies, the LIMIT, of THE AVERAGE RATE OF CHANGE.

27. The Characteristic Feature of Calculus

Every POLYNOMIAL FUNCTION $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_2 x^2 + a_1 x^1 + a_0$ is CONTINUOUS, and every RATIONAL FUNCTION $H(x) = \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomial functions, is CONTINUOUS on the set of all points at which $g(x) \neq 0$.

This implies that *the properties of limits lead to corresponding properties of continuity*. Thus, a function f is defined as continuous, if the LIMIT is a REAL NUMBER. So $f(x) = \sqrt{4 - x^2}$ is not continuous at 3 because $f(3)$ is not DEFINED since $f(3) = \sqrt{4 - 9} = \sqrt{-5} = y^2 = -5$. There's no such thing as $f(3) = -5$!

If f is a function, then we say (A) is the limit of $f(x)$ as (x) approaches (a). We already know the notation as the $\lim_{x \rightarrow 0} f(x) = A$.

In a RIGOROUS DEVELOPMENT OF MATHEMATICS, a FUNCTION is defined as the SET OF ORDERED PAIRS, such that if $[x, y]$ and $[x, z]$ are the set of f , then $y = z$. This is so because for the function $g(x) = x^2$ if $2 \leq x \leq 4$

$= x + 1$ if $1 \leq x \leq 2$, the basis of finding the DOMAIN of (g) is in

DETERMINING THE INTERVAL $[1, 4]$.

A function $y = f(x)$ has (y) as the VALUE and $f(x)$ as the ARGUMENT. Thus the quantity (y) is determined by the value of (x). Thus we indicate the dependence of (x) by means of the FORMULA $y = f(x)$. This implies that the GRAPH OF FUNCTION f , is defined to be the graph of the EQUATION $y = f(x)$. For instance, the formula $g(x) = 2x + 3$ defines the function (g). Thus, the graph of this function is the straight line with the slope $m = 2$ and y - intercept 3. This results in the EQUATION OF THE LINE $y = mx + b$.

So the SET OF ALL REAL NUMBERS, is both the DOMAIN and RANGE of (g) . Suppose (ϵ) is any positive number, then there is some positive number (δ) , such that the $|f(x) - A|$ is $< \epsilon$ if $0 < |x - x_0| < \delta$. This verbal statement is EXPRESSED in terms of an INEQUALITY.

So, the symbolic form of the statement $\lim_{x \rightarrow x_0} f(x) = A$ is so far attached to a CLEAR MEANING to the statement, $f(x)$ approaches (A) or tends to the LIMIT (A) , as (x) approaches x_0 , and thus (A) is understood to stand for some particular REAL NUMBER.

Therefore, the CHARACTERISTIC FEATURE OF CALCULUS is its use of the LIMITING PROCESS. Thus, both DIFFERENTIATION and INTEGRATION involve the notion of passage to a LIMIT.

If we think of (x) as a point on a number scale, say the x – axis, then the $|x|$ is the numerical distance, and always non-negative between (x) and the ORIGIN. If (y) is a symbol for the value of the function f , at (x) , we write $y = f(x)$. So (y) is the DEPENDENT VARIABLE. In common word, we say that (y) is a function (x) .

The subject of CALCULUS, rests upon the REAL NUMBER SYSTEM, and the THEORY OF LIMITS. We can thus define any FUNCTION in the sense of any *determinate correspondence between two classes of objects*.

28. The Geometric Picture Of Calculus

The ORDER of a DIFFERENTIAL EQUATION is the order of the DERIVATIVE of the highest order appearing in it.

For instance, $\frac{\partial^2 y}{\partial x^2} + 2 \frac{\partial y}{\partial x} + 3y - 7 \sin x + 4x = 0$ can be written as $y'' + y' + 3y - 7 \sin x + 4x = 0$. This EQUATION is said to be of the order two. For $\partial y = (x + 2) \partial x$ is of order one. These are the definitions of the DIFFERENTIAL EQUATIONS of second order and first respectively.

Note if $f(0) = 1$, then $f(x) = e^x$. So we can derive the notation for the diverging SEQUENCE 1, 2, 3, 4,, n as $n^{e^x(k-1)}$ where $|e^x - k| < 1$ for all (n) .

The $\lim_{(x,y) \rightarrow [0,0]} \frac{x^2 - y^2}{x^2 + y^2}$ DOES NOT EXIST since the LIMIT is 1 along the x – axis and -1 along the y – axis. Hence, there is no common limit as one approaches $[0, 0]$.

We then note that PARTIAL DERIVATIVES EXIST, when the required LIMIT exists. Thus, the function $w = (x, y, z, \dots)$ of three or more INDEPENDENT VARIABLES may be defined, but with no GEOMETRICAL PICTURE. Therefore, CALCULUS is limited to the FUNCTIONS OF TWO VARIABLES.

Let $f(x, y) = x^2 \sin(y)$, then $f_x(x, y) = 2x \sin y$ and $f_y(x, y) = x^2 \cos y$. This implies that if (x) varies while (y) is HELD FIXED, f becomes a function of (x) . In other words, the derivative with respect to (x) is the $\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$ and so represents the 1st DERIVATIVE of f with respect to (x) .

In general, this is denoted as $f_x(x, y)$ or $\frac{\partial f}{\partial x}$. Similarly, the converse holds. That is, when $f(y)$ is computed, (x) is temporarily treated like a constant.

In general, if (a) and (b) are in the INTERVAL OF CONVERGENCE, then $\int_a^b f(x) dx = \sum_{n=0}^{+\infty} \left[\frac{a_n (x-C)^{n+1}}{n+1} \right]_{x=b}^{x=a}$. This is the ANTIDERIVATIVE of f' and is thus the interval of convergence of the POWER SERIES which is the same as for the original power series f .

For instance, let f be the function defined by a POWER SERIES $\sum_{n=0}^{+\infty} a_n (x-C)^{n+1}$ on the INTERVAL OF CONVERGENCE with RADIUS R_1 . Then the INTEGRAL $\int f(x) dx = \sum_{n=0}^{+\infty} \frac{a_n (x-C)^{n+1}}{n+1} + K$ for $|x-C| < R_1$. Thus, the DERIVATIVE f' , is the INTERVAL OF CONVERGENCE of the POWER SERIES, and is the same as for the original power series f .

So, if f is a FUNCTION defined by a power series $\sum_{n=0}^{+\infty} n a_n (x-C)^{n-1}$ on its interval of convergence, with radius of convergence R_1 , then f is DIFFERENTIABLE on that INTERVAL as $f'(x) = \sum_{n=0}^{+\infty} n a_n (x-C)^{n-1} + K$ for $|x-C| < R_1$.

The notion of SEQUENCES should be clear following STANDARD ARITHMETIC OPERATIONS. For instance, the sequence $[(-1)^n]$ implies $-1, 1, -1, 1, \dots$ and so $|(-1)^n| < 1$ for all (n) . The sequence is thus interpreted as its values oscillating between 1 and -1.

Take n_0 to be the smallest positive INTEGER greater than $\frac{1}{\epsilon}$. Then $\frac{1}{\epsilon} < n_0$. Hence, if $n > n_0$, then $n > \frac{1}{\epsilon}$ and $\frac{1}{n} < \epsilon$. So if $n \geq n_0$, then the $\left| \frac{1}{n} - 0 \right| < \epsilon$. This is the PROOF that $\frac{1}{n}$ is CONVERGENT. We thus write that the $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$.

Note that if $n = \frac{1}{\epsilon}$, then $\epsilon(n) = 1$ and so $\epsilon = \frac{1}{n}$.

We then can say that the SEQUENCE $2(n)$ is DIVERGENT, since the $\lim_{n \rightarrow \infty} 2n \neq L$.

Thus:

$2n$ is the sequence 2, 4, 6, 8, ..., $2n$, ...

n^2 is the sequence 1, 4, 9, 16, 25, ..., n^2 , ...

$2n-1$ is the sequence 1, 3, 5, 7, 9, ...

So S_n converges to L if the $\lim_{n \rightarrow \infty} S_n = L$ and, if S_n is not convergent, then it is divergent.

The $\lim_{n \rightarrow \infty} S_n = L$ implies that for any positive REAL NUMBER $\epsilon > 0$, there exists a positive INTEGER n_0 such that when $n > n_0$, we have the $|S_n - L| < \epsilon$. Note that S_n is indicated by the first few terms as $S_1, S_2, S_3, \dots, S_n, \dots$

In summary, an INFINITE SEQUENCE is a FUNCTION whose DOMAIN is the SET of positive INTEGERS, where S_n is the VALUE.

EPILOG

Mathematics is a subject of intellectual construction. Its knowledge, is the presence of nobility.



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